LIMIT OF A FUNCTION

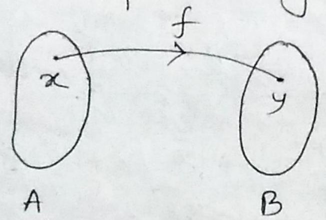
Previous Part Discussion :→

Function :→ For two non-empty sets A and B, a function f from A to B is a subset of A×B such that for every element of the set A, there exists an unique element on the set B. It is denoted by $f: A \to B$.

The first set A is called the domain of f and is denoted by $D(f)$ or D_f and the second set B is called the co-domain of f and is denoted by codom.

If an element x of the set A is related to an element y of the set B, then we write $y = f(x)$.

Here y is called the image of x and x is called the pre-image of y.



$f(x) = y$.

Range of a fuction :→ Let, $f: A \to B$ be a function. The range of the function f denoted by $R(f)$ or R_f, is a subset of B and defined by

$R(f) = \{f(x) : x \in A\}$.

Example :→ i) Consider the function $f(x)$ as $y = f(x) = x^2 - 3x + 2$.

It is defined for all real values of x in $(-\infty, \infty)$.

ii) Consider the function given by $y = f(x) = \frac{x}{x^2 - 1}$.

This function is defined for all real values of x except $x = \pm 1$.

Real Valued Function :→ A function f is called a real valued function if the range set of f is a subset of R (the set of real numbers).

Bounded Function :→ Let, $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function, f is said to be bounded above on D if there exists a real number M such that $f(x) \leq M, \forall x \in D$. M is said to be an upper bound of f on D.

f is said to be bounded below on D if there exists a real number m such that $f(x) \geq m, \forall x \in D$. m is said to be a lower bound of f on D.

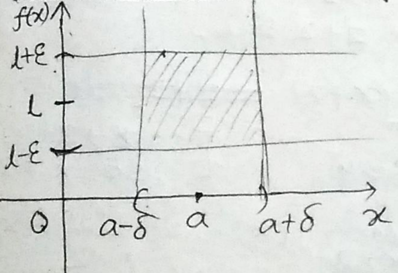
f is said to be bounded on D if f is bounded above as well as bounded below on D.

Example :→ $f(x) = \sin x, x \in \mathbb{R}$. Since $-1 \leq \sin x \leq 1, \forall x \in \mathbb{R}$ so f is bounded on $\mathbb{R}$.

Limit of a function :→ Let, $f(x)$ be a real-valued function defined on $[a,b]$. Then the function $f(x)$ is said to tend to an real number l as $x \to a$ if for any positive number ε (however small) $\exists$ a positive number δ s.t. $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$.

And we write $\lim\limits_{x \to a} f(x) = l$.

Geometric Interpretation :→ If the function $f(x)$ approaches to l then we can find a neighbourhood of the point a, $(a-\delta, a+\delta) \setminus \{a\}$ s.t. for all x in that neighbourhood, the image $f(x)$ lie in $(l-\varepsilon, l+\varepsilon)$.



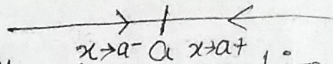
⊛ $\lim\limits_{x \to a} f(x) = \infty$ means for any +ve number G (however large), $\exists$ a +ve number δ s.t. $f(x) > G$ whenever $0 < |x - a| < \delta$.

⊛ $\lim\limits_{x \to \infty} f(x) = l$ means for any $\varepsilon > 0$, $\exists$ a +ve number G (however large) s.t. $|f(x) - l| < \varepsilon$ whenever $x > G$.

⊛ $\lim\limits_{x \to -\infty} f(x) = l$ means for any $\varepsilon > 0$, $\exists$ a +ve number G (however large) s.t. $|f(x) - l| < \varepsilon$ whenever $x < -G$.

Right hand and Left hand Limits :→ For $x \to a+$, if $\lim_{x \to a+} f(x) = l_1$, then we say that the function $f(x)$ possesses right hand limit l_1 as $x \to a+$.

Similarly, for $x \to a-$, if $\lim_{x \to a-} f(x) = l_2$, then we say that the function $f(x)$ possesses left hand limit l_2 as $x \to a-$.



Existance of Limit :→ The limit of a function $f(x)$ as $x \to a$ exists if

i) both the limits $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist,

and ii) $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$.

Then $\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$.

Otherwise we say that the limit does not exist.

Example :→ 1. Find the limit of the function $f(x)$, where $f(x) = \frac{x^2-4}{x-2}$ as $x \to 2$.

Solution :→

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} \frac{x^2-4}{x-2} = \lim_{x \to 2+} (x+2) \quad [\because x \to 2+, \text{ so } x \neq 2]$$
$$= 2+2 = 4.$$

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \frac{x^2-4}{x-2} = \lim_{x \to 2-} (x+2) \; \cancel{2+2=4} \quad [\because x \to 2-, \text{ so } x \neq 2]$$
$$= 2+2 = 4$$

Since $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x)$, so $\lim_{x \to 2} f(x)$ exists and $\lim_{x \to 2} \left(\frac{x^2-4}{x-2}\right) = 4$.

2. Show that $\lim_{x \to 0} \sqrt{x}$ does not exist.

Solution :→ Let, $f(x) = \sqrt{x}$.

$\therefore \lim_{x \to 0+} f(x) = \lim_{x \to 0+} \sqrt{x} = 0$.

Now $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \sqrt{x}$, does not exists $[\because x < 0]$

Thus $\lim_{x \to 0} \sqrt{x}$ does not exist.